The Derived Model Theorem II

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This paper is based on notes on lectures given by Woodin in the Berkeley set theory seminar.

Theorem 1 (Derived model theorem II, Woodin). *Let λ be a limit of Woodin cardinals. Let G be V-generic over $Col(\omega, <\lambda)$. Let $V(\mathbb{R}^*_G) = HOD^{(V[G],V)}_{V \cup \mathbb{R}^*_G \cup \{\mathbb{R}^*_G\}}$ and*

$$\mathbb{R}^*_G = \bigcup_{\alpha<\lambda} \mathbb{R} \cap V[G \restriction \alpha],$$

$$\begin{aligned} \mathrm{Hom}^*_G = \{A \subset \mathbb{R}^*_G : \exists \alpha < \lambda \exists T, U \in V[G\restriction\alpha](A = p[T] \cap \mathbb{R}^*_G \\ \wedge V[G\restriction\alpha] \models T, U \text{ are } < \lambda \text{ complementing trees})\}, \end{aligned}$$

$$\mathcal{A}_G = \{B \subset \mathbb{R}^*_G : B \in V(\mathbb{R}^*_G) \text{ and } L(B, \mathbb{R}^*_G) \models AD^+\}.$$

Then

(1) For $B, C \in \mathcal{A}_G$, either $L(B, \mathbb{R}^*_G) \subset L(C, \mathbb{R}^*_G)$ or $L(C, \mathbb{R}^*_G) \subset L(B, \mathbb{R}^*_G)$.

(2) $L(\mathcal{A}_G, \mathbb{R}^*_G) \models AD^+$.

(3) For each $B \in \mathcal{P}(\mathbb{R}^*_G) \cap V(\mathbb{R}^*_G)$, the following are equivalent

- *(i) B is Suslin-co-Suslin in $V(\mathbb{R}^*_G)$.*
- *(ii) $B \in \mathcal{A}_G$ and B is Suslin-co-Suslin in $L(\mathcal{A}_G, \mathbb{R}^*_G)$.*
- *(iii) $B \in \mathrm{Hom}^*_G$.*

Remark. The model $L(\mathcal{A}_G, \mathbb{R}^*_G)$ is called the new derived model. By results of [2], it contains the old derived model $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)$. Steel observed that if λ is a limit of $<\lambda$-strongs, then $\mathcal{A}_G = \mathrm{Hom}^*_G$.

Question. If $\mathcal{P}(\mathbb{R}^*_G) \cap L(\text{Hom}^*_G, \mathbb{R}^*_G) \subset \text{Hom}^*_G$, must $\mathcal{A}_G = \text{Hom}^*_G$? The answer is negative if $L(\mathcal{A}_G, \mathbb{R}^*_G) \models$ "the largest Suslin cardinal is some θ_α".

We show (3) of the theorem first. (ii)→(i) is clear. (i)→(iii) follows as in [2], since any tree $T \in V(\mathbb{R}^*_G)$ is in some $\text{Hom}^{V[G|\alpha]}_{<\lambda}$, $\alpha < \lambda$. (iii)→(ii) is in [2].

Now we start proving (1), that is, all sets in $\mathcal{A}_G$ have compatible Wadge degrees, witnessed by continuous functions coded in $\mathbb{R}^*_G$.

Given any $B \in \mathcal{A}_G$, we record some facts from [1]. Since $L(B, \mathbb{R}^*_G) \models \text{AD}^+$, $L(B, \mathbb{R}^*_G)$ has a largest Suslin cardinal κ_B. Let $\boldsymbol{\Gamma}$ be the pointclass consisting of κ_B-Suslin sets. So $\boldsymbol{\Gamma}$ is closed under $\exists^{\mathbb{R}}, \forall^{\mathbb{R}}$ and has the scale property. Let Γ be an associated lightface pointclass and T_B be the tree on $\omega \times \omega \times \omega \times \kappa_B$ for the scale on the universal Γ set $U \subset \omega \times \mathbb{R} \times \mathbb{R}$. Then for any $x \in \mathbb{R}^*_G$, T_B certifies a Γ-good wellordering $<^T_x$ of $C_\Gamma(x) = \mathbb{R}^{L[T_B, x]}$. To be precise, there is a natural number k such that $(k, x, y) \in U \leftrightarrow y$ codes a $<^{T_B}_x$-initial segment of $\mathbb{R}^{L[T_B, x]}$.

We may consider the Turing degree ultrapower $\prod_{\mathcal{D}} \text{Ord} / \mu_M$ in $L(B, \mathbb{R}^*_G)$. This is taken from all functions $f \in L(B, \mathbb{R}^*_G)$ where $dom(f) = \mathcal{D}$ and $f(d) \in \text{Ord}$ for each d, μ_M is the cone measure on Turing degrees. Let $T^*_B = \prod_{\mathcal{D}} T/\mu_M$ be the tree on $\omega \times \omega \times \omega \times \kappa^*_B$ in the ultrapower. Then $L(B, \mathbb{R}^*_G) = L(T^*_B, \mathbb{R}^*_G)$.

Let $\mathbb{P}$ be recursively pointed Sacks forcing. Suppose g is $V(\mathbb{R}^*_G)$-generic for $\mathbb{P}$. Then letting x_g be the Sacks real associated to g, . We know

(i) g naturally induces an $L(B, \mathbb{R}^*_G)$-ultrafilter on $\mathcal{P}(\mathbb{R}^*_G)^{L(B, \mathbb{R}^*_G)}$. The ultraproduct $\prod_{x \in \mathbb{R}^*_G} L[T_B, x]/g$ agrees with $\prod_{\mathcal{D}} \text{Ord} / \mu_M$ using functions in $L(B, \mathbb{R}^*_G)$ on ordinals, namely, the mapping $[f] \mapsto [f']$, defined by $f'(x) = f([x]_T)$, is a well-defined bijection from $\prod_{\mathcal{D}} \text{Ord} / \mu_M$ to $\prod_{x \in \mathbb{R}^*_G} \text{Ord} / g$.

(ii) $(\prod_{x \in \mathbb{R}^*_G} L[T_B, x])/g = L[T^*_B, x_g]$, and $T^*_B = (\prod_{x \in \mathbb{R}^*_G} T_B)/g$, $x_g = (\prod_{x \in \mathbb{R}^*_G} x)/g$.

(iii) $(T^*_B)_{x_g}$ certifies a wellordering $<^{T^*_B}_{x_g}$ of the reals in $L[T^*_B, x_g]$, in the following sense:

$\exists w (k, x_g, z, w) \in [T^*_B] \leftrightarrow z$ codes an $<^{T^*_B}_{x_g}$-initial segment of the reals in $L[T^*_B, x_g]$.

(iv) all reals in $\mathbb{R}^*_G$ are Turing reducible to x_g.

(v) for any $B_0 \in L(B, \mathbb{R}^*_G)$, there is $B^*_0 \in L[T^*_B, x_g]$ such that $B^*_0 \cap \mathbb{R}^*_G = B_0$. So there is a real $z \in L[T^*_B, x_g]$ which codes B_0 relative to $\mathbb{R}^*_G$ in the following sense:

For all $l \in \omega$, if $\{l\}^{x_g} \in \mathbb{R}^*_G$, then $\{l\}^{x_g} \in B_0 \leftrightarrow l \in z$.

Here $\{l\}^x$ is the lth real recursive in x.

Suppose toward a contradiction that there are $B, C \in \mathcal{A}_G$, but neither $L(B, \mathbb{R}^*_G) \subset L(C, \mathbb{R}^*_G)$ nor $L(C, \mathbb{R}^*_G) \subset L(B, \mathbb{R}^*_G)$. We have $T^*_B, T^*_C, \mathbb{P}$ as above. So whenever g is generic for $\mathbb{P}$, In $V(\mathbb{R}^*_G)[g]$, there are $B_0 \in L(B, \mathbb{R}^*_G), C_0 \in L(C, \mathbb{R}^*_G)$ such that $(B_0, \mathbb{R}^*_G \setminus B_0)$ and $(C_0, \mathbb{R}^*_G \setminus C_0)$ represent incompatible Wadge degrees using continuous functions coded in $\mathbb{R}^*_G$. We say that B_0, C_0 witness Wadge incompatibility in this case. According to (v) as above, pick β_0, γ_0 so that the β_0th real in $<^{T^*_B}_{x_g}$ codes B_0 relative to x_g and $\mathbb{R}^*_G$, the γ_0th real in $<^{T^*_C}_{x_g}$ codes C_0 relative to x_g and $\mathbb{R}^*_G$. We may further assume that β_0, γ_0 is least. To be precise, $(\beta_0, \gamma_0, B_0, C_0)$ is the unique tuple such that $\psi(\beta_0, \gamma_0, B_0, C_0, T^*_B, T^*_C, x_g, \mathbb{R}^*_G)$ holds, where $\psi(\beta, \gamma, X, Y, T^*_B, T^*_C, x_g, \mathbb{R}^*_G)$ is the conjunction of the following formulas

(i) β, γ is the least β', γ' in Gödel ordering such that:
There are $B' \in L(T^*_B, \mathbb{R}^*_G)$ and $C' \in L(T^*_C, \mathbb{R}^*_G)$ such that the β'th real in $<^{T^*_B}_{x_g}$ codes B' relative to x_g and $\mathbb{R}^*_G$, the γ'th real in $<^{T^*_C}_{x_g}$ codes C' relative to x_g and $\mathbb{R}^*_G$, B', C' witness Wadge incompatibility

(ii) X is coded by the βth real in $<^{T^*_B}_{x_g}$ relative to x_g and $\mathbb{R}^*_G$,

(iii) Y is coded by the βth real in $<^{T^*_C}_{x_g}$ relative to x_g and $\mathbb{R}^*_G$,

Fix the unique witness $(\beta_0, \gamma_0, B_0, C_0)$, and assume that the fact is forced by some p_0:

$$V(\mathbb{R}^*_G) \models p_0 \Vdash_{\mathbb{P}} (\beta_0, \gamma_0, B_0, C_0) \text{ is the unique tuple such that } \psi(\check{\beta}_0, \check{\gamma}_0, \check{B}_0, \check{C}_0, \check{T}^*_B, \check{T}^*_C, \dot{x}_g, \check{\mathbb{R}}^*_G) \text{ holds.}$$

Here $\dot{x}_g$ is the canonical name for the Sacks real.

We may assume $p_0, T^*_B, T^*_C \in V$ by working in some intermediate extension. Let $\phi_1(v_0, v_1, p_0, x)$ be the sentence

There are β', γ', B', C' such that $p_0 \Vdash_{\mathbb{P}} \psi(\check{\beta}', \check{\gamma}', \check{B}', \check{C}', \check{v}_0, \check{v}_1, \dot{x}_g, \check{\mathbb{R}})$ and $x \in B'$.

$\phi_2(v_0, v_1, p_0, x)$ is defined similarly, with "$x \in B'$" replaced by "$x \in C'$".

The next lemma is of no direct use, but helps understanding the meanings of ϕ_1 and ϕ_2.

Lemma 2. *For $x \in \mathbb{R}^*_G$,*

$$x \in B_0 \leftrightarrow V(\mathbb{R}^*_G) \models \phi_1(T^*_B, T^*_C, p_0, x);$$
$$x \in C_0 \leftrightarrow V(\mathbb{R}^*_G) \models \phi_2(T^*_B, T^*_C, p_0, x).$$

Proof. $\rightarrow$ Since $\beta_0, \gamma_0, B_0, C_0$ is a witness to ϕ_1.

$\leftarrow$ In fact, $\beta_0, \gamma_0, B_0, C_0$ is the unique witness to ϕ_1. □

We will show that $\phi_1(T^*_B, T^*_C, p_0, x)$ and $\phi_2(T^*_B, T^*_C, p_0, x)$ defines $\mathrm{Hom}^V_{<\lambda}$ sets over V from the tree production lemma [2], and this will be a contradiction.

Lemma 3. *Let G be $\mathbb{Q}_{<\lambda}$ generic over V. G then induces generic ultrapower embeddings $j_\delta : V \rightarrow M_\delta = Ult(V, G \cap V_\delta)$ and their direct limit $j^* : V \rightarrow M = \mathrm{dirlim}_{\delta<\lambda} M_\delta$. Suppose that $\mathbb{R} \cap M = \mathbb{R}^*_G$ and that $\beta_0, \gamma_0 \in \mathrm{wfp}(M)$. Let g be generic over $V[G]$ for the forcing $(\mathbb{P})^{\mathbb{R}^*_G}$, i.e. forcing with recursively pointed perfect trees coded in $\mathbb{R}^*_G$. Let x_g be the Sacks real of g. Then*

(i) In $M[x_g]$, let $<^{j^*(T^*_B), \max(\beta_0,\gamma_0)+1}_{x_g}$ be the wellorder certified by $j^*(T^*_B)$ up to length $\max(\beta_0, \gamma_0) + 1$ in the following sense:

$\exists w(k, x, y, w) \in [j^*(T^*_B)] \leftrightarrow y$ codes a wellordered initial segment of $<^{j^*(T^*_B), \max(\beta_0,\gamma_0)+1}_{x_g}$ of length at most $\max(\beta_0, \gamma_0) + 1$.

Then $<^{j^*(T^*_B), \max(\beta_0,\gamma_0)+1}_{x_g}$ is a well-defined wellorder (in V[G][g]).

And similarly for $<^{j^*(T^*_C), \max(\beta_0,\gamma_0)+1}_{x_g}$.

(ii) $<^{j^*(T^*_B), \max(\beta_0,\gamma_0)+1}_{x_g}$ is an initial segment of $<^{T^*_B}_{x_g}$;

$<^{j^*(T^*_C), \max(\beta_0,\gamma_0)+1}_{x_g}$ is an initial segment of $<^{T^*_C}_{x_g}$;

(iii) For $z \in \mathbb{R}^{L[T^*_B, x_g]}$ among the first $\max(\beta_0, \gamma_0)$ reals in the order $<^{T^*_B}_{x_g}$, if z codes a set of reals $E \subset \mathbb{R}^*_G$ relative to x_g and $\mathbb{R}^*_G$, then $E \in L(T^*_B, \mathbb{R}^*_G)$ iff $E \in L(j^*(T^*_B), \mathbb{R}^*_G)$;

For $z \in \mathbb{R}^{L[T^*_C, x_g]}$ among the first $\max(\beta_0, \gamma_0)$ reals in the order $<^{T^*_C}_{x_g}$, if z codes a set of reals $E \subset \mathbb{R}^*_G$ relative to x_g and $\mathbb{R}^*_G$, then $E \in L(T^*_C, \mathbb{R}^*_G)$ iff $E \in L(j^*(T^*_C), \mathbb{R}^*_G)$.

Proof. We show statements about B.

(i) Let $S_{T_B^*} \in L[T_B^*]$ be the tree of on $\omega \times \omega \times \omega \times \kappa_B^* \times \kappa_B^*$ attempting (x, y_1, y_2, w_1, w_2) with $(k, x, y_1, w_1), (k, x, y_2, w_2) \in [T_B^*]$ and either y_1, y_2 incompatible or y_1 does not code a wellorder. Then $S_{T_B^*}$ is wellfounded, since $S_{T_B^*} = \prod S_{T_B}/g$ in the ultraproduct and each S_{T_B} is wellfounded. (Here S_{T_B} is defined from T_B in the same way as $S_{T_B^*}$ is defined from T_B^*.) So by elementarity $j^*(S_{T_B^*}^*)$ is a wellfounded tree on $\omega \times \omega \times \omega \times j^*(\kappa_B^*) \times j^*(\kappa_B^*)$ from the point of view of M. Therefore we may define in $M[x_g]$ the wellorder $<_{x_g}^{j^*(T_B^*)}$. $<_{x_g}^{j^*(T_B^*)}$ may not be a wellorder in the real universe, but since β_0, γ_0 are in the wellfounded part of M, the length $\max(\beta_0, \gamma_0) + 1$ initial segment of that is a true wellorder. In other words, $<_{x_g}^{j^*(T_B^*), \max(\beta_0, \gamma_0)+1}$ is a true wellorder.

(ii) Suppose otherwise. Then there is $p \in \mathbb{R}_G^*$, $\alpha \leq \max(\beta_0, \gamma_0)$ and n such that for instance,

$$V(\mathbb{R}_G^*) \models p \Vdash_{\mathbb{P}} \text{“}n \text{ is in the } \alpha\text{th real of } <_{x_g}^{T_B^*}\text{”},$$

$$M \models p \Vdash_{\mathbb{P}} \text{“}n \text{ is not in the } \alpha\text{th real of } <_{x_g}^{j^*(T_B^*)}\text{”}.$$

Pick a Woodin cardinal $\delta < \lambda$ large enough so that $p \in M_\delta$ and $\alpha \in ran(j_\delta^*)$. Here $j_\delta^* : M_\delta \to M$ is the tail of the direct limit embedding. Suppose $j_\delta^*(\beta) = \alpha$. By elementarity,

$$M_\delta \models p \Vdash_{\mathbb{P}} \text{“}n \text{ is not in the } \beta\text{th real of } <_{x_g}^{j_\delta(T_B^*)}\text{”}.$$

In M_δ, let $\dot{y}, \dot{w}$ be $\mathbb{P}$-names so that the following is forced by p: “$(k, \dot{x}_g, \dot{y}, \dot{w}) \in [j_\delta(T_B^*)]$, certifying that $\dot{y}$ codes a length $\beta + 1$ initial segment of $<_{\dot{x}_g}^{j_\delta(T_B^*)}$, and n is not in the βth real”. Hence in M, the following is forced by p: “$(k, \dot{x}_g, j_\delta^*(\dot{y}), j_\delta^*(\dot{w})) \in [j^*(T_B^*)]$, certifying that $j^*(\dot{y})$ codes a length $\alpha + 1$ initial segment of $<_{\dot{x}_g}^{j^*(T_B^*)}$, and n is not in the αth real‘”.

It is a basic property of recursively pointed Sacks forcing that every countable set of ordinals in the generic extension is covered by a countable set in the ground model. Hence in M_δ there exists $q \leq_{\mathbb{P}} p$ and a countable set of ordinals a such that $q \Vdash_{\mathbb{P}} ran(\dot{w}) \subset \check{a}$. Hence in M, $q \Vdash_{\mathbb{P}} ran(j_\delta^*(\dot{w})) \subset j_\delta^*(\check{a})$. But $j_\delta^*(a) = j_\delta^{*\prime\prime} a$ since a is countable in M_δ! Now let h be $(\mathbb{P})^{\mathbb{R}_G^*}$-generic over $V[G]$ such that $q \in h$. Then $ran(j_\delta^*(\dot{w}))_h \subset j_\delta^{*\prime\prime} a \subset ran(j_\delta^*)$, which will be important in the following argument.

Let U be the tree in $V[G]$ attempting to build (x, y, y', w, w') with $(k, x, y, w) \in [T_B^*]$, $(k, x, y', w') \in [j_\delta(T_B^*)]$, and y incompatible with y'. U is then illfounded in $V[G]$ since it has a branch $(x_h, y, (j^*_\delta(\dot{y}))_h, w, (j^*_\delta)^{-1''}(j^*_\delta(\dot{w}))_h) \in [j^*(T^*_B)]$ in $V[G][h]$: Here $(k, x_h, y, w) \in [T^*_B]$ is any branch certifying that n is in the αth real of $<^{T^*_B}_{x_h}$; $(k, x_h, (j^*_\delta(\dot{y}))_h, (j^*_\delta(\dot{w}))_h) \in [j^*(T^*_B)]$ certifies that n is not in the αth real of $<^{j^*(T^*_B)}_{x_h}$, and when pulled back by j^*_δ, we get $(x_h, (j^*_\delta(\dot{y}))_h, (j^*_\delta)^{-1''}(j^*_\delta(\dot{w}))_h) \in [j_\delta(T^*_B)]$ as in definition of U.

However U must be wellfounded! If (x, y, w, y', w') were a branch of U in $V[G]$, then $(k, x, y, j_\delta{}''w)$ and (k, x, y', w') are in $[j_\delta(T^*_B)]$, with y incompatible with y'. Since M_δ is fully wellfounded, by absoluteness, $S_{j_\delta(T^*_B)}$ is illfounded in M. Hence $S_{T^*_B}$ is illfounded in V, which is absurd.

(iii) We show $\leftarrow$. The other direction is similar. Suppose the αth real in $<^{T^*_B}_{x_g}$ codes $E \in L(j^*(T_B), \mathbb{R}^*_G)$ relative to x_g and $\mathbb{R}^*_G$, $\alpha \leq \max(\beta_0, \gamma_0)$. Let q_0 force this fact:

$$L[j^*(T^*_B), \mathbb{R}^*_G] \models q_0 \Vdash_{\mathbb{P}} \text{“the } \check{\alpha}\text{th real certified by } (j^*(T^*_B))\check{}\ \text{codes } \check{E} \text{ relative } \dot{x}_g \text{ and } \check{\mathbb{R}^*_G}\text{”}.$$

Then for $x \in \mathbb{R}^*_G$,

$$\begin{aligned} x \in E \leftrightarrow & \text{Whenever } h \text{ is Sacks generic over } V[G] \text{ such that } q_0 \in h, \\ & L(j(T^*_B), \mathbb{R}^*_G)[h] \models \text{“Let } z \text{ be the } \alpha\text{th real in } <^{j^*(T^*_B)}_{x_h} \\ & \quad \text{then for any number } l \text{ such that } x = \{l\}^{x_h}, l \in z\text{”} \\ \leftrightarrow & \text{Whenever } h \text{ is Sacks generic over } V[G] \text{ such that } q_0 \in h, \\ & L(T^*_B, \mathbb{R}^*_G)[h] \models \text{“Let } z \text{ be the } \alpha\text{th real in } <^{T^*_B}_{x_h} \\ & \quad \text{then for any number } l \text{ such that } x = \{l\}^{x_h}, l \in z\text{”} \\ \leftrightarrow & L(T^*_B, \mathbb{R}^*_G) \models q_0 \Vdash_{\mathbb{P}} \text{“ let } l \text{ be any number such that } \check{x} = \{l\}^{\dot{x}_g}, \\ & \quad \text{then } l \text{ is in the } \check{\alpha}\text{th real in } <^{\check{T^*_B}}_{\dot{x}_g}\text{”}. \end{aligned}$$

The second equivalence follows from (ii). So $E \in L(T^*_B, \mathbb{R}^*_G)$. □

Lemma 4. *Let G, j^*, M be as in the hypothesis of Lemma 3. Suppose h is $(\mathbb{P})^{\mathbb{R}^*_G}$-generic over $V[G]$ so that $p_0 \in h$. Then*

$$\begin{aligned} M[x_h] \models & (\beta_0, \gamma_0, B_0, C_0) \textit{ is the only tuple so that} \\ & \psi(\beta_0, \gamma_0, B_0, C_0, j^*(T^*_B), j^*(T^*_C), x_h, \mathbb{R}^*_G) \textit{ holds,} \end{aligned}$$

Proof. We know in $V(\mathbb{R}^*_G)[x_h]$, $(\beta_0, \gamma_0, B_0, C_0)$ is the only tuple so that $\psi(\beta_0, \gamma_0, B_0, C_0, j^*(T^*_B), j^*(T^*_C), x_g, \mathbb{R}^*_G)$ holds.

By Lemma 3 (i)(ii), for all $\beta' \le \max(\beta_0, \gamma_0)$, the β'th real in $<_{x_h}^{j^*(T_B^*)}$ (or $<_{x_h}^{j^*(T_C^*)}$) is equal to the β'th real in $<_{x_h}^{T_B^*}$(or $<_{x_h}^{T_C^*}$). So in $M[x_h]$, the β_0th in $<_{x_h}^{T_B^*}$ codes B_0 relative to $x_h, \mathbb{R}_G^*$, the γ_0th in $<_{x_h}^{T_C^*}$ codes C_0 relative to $x_h, \mathbb{R}_G^*$, and B_0, C_0 witness Wadge incompatibility. It remains to see that $B_0 \in L(j^*(T_B^*), \mathbb{R}_G^*)$, $C_0 \in L(j^*(T_C^*), \mathbb{R}_G^*)$, and if a pair of reals in some smaller position β', γ' code B', C' in $L(j^*(T_B), \mathbb{R}_G^*)$, $L(j^*(T_C), \mathbb{R}_G^*)$ respectively, then they are actually in $L(T_B^*, \mathbb{R}_G^*)$, $L(T_C^*, \mathbb{R}_G^*)$ respectively. This is exactly what we get from Lemma 3(iii). The lemma follows. □

Corollary 5. *Let G, j^*, M be as in the hypothesis of Lemma 3. Then for $x \in \mathbb{R}_G^*$,*

$$x \in B_0 \leftrightarrow M \models \phi_1(j^*(T_B^*), j^*(T_C^*), p_0, x);$$
$$x \in C_0 \leftrightarrow M \models \phi_2(j^*(T_B^*), j^*(T_C^*), p_0, x).$$

Lemma 6. *(1) Whenever $K \in V(\mathbb{R}_G^*)$ is $< \lambda$-generic over V, x is a real in $V[K]$,*

$$x \in B_0 \leftrightarrow V[K] \models \phi_1(T_B^*, T_C^*, p_0, x),$$
$$x \in C_0 \leftrightarrow V[K] \models \phi_2(T_B^*, T_C^*, p_0, x).$$

(2) Whenever $H \in V(\mathbb{R}_G^*)$ is $\mathbb{Q}_{<\delta}$-generic over V, $\delta < \lambda$ is Woodin, letting $j_H : V \to Ult(V, H) \subset V[H]$ be the stationary tower embedding, we have

$$x \in B_0 \leftrightarrow Ult(V, H) \models \phi_1(j_H(T_B^*), j_H(T_C^*), p_0, x);$$
$$x \in C_0 \leftrightarrow Ult(V, H) \models \phi_2(j_H(T_B^*), j_H(T_C^*), p_0, x).$$

Proof. (1) As in [2], we may construct $j' : V[K] \to M \subset V[G]$ as the direct limit of stationary tower embeddings such that (G, j', M) satisfies the hypothesis of Lemma 3 with V replaced by $V[K]$. Apply Corollary 5 to j', and then pull back to $V[K]$ via j'.

(2) Let $j' : Ult(V, H) \to M \subset V[G]$ be the canonical embedding so that $(G, j' \circ j_H, M)$ satisfies Lemma 3. Apply Corollary 5 to $j' \circ j_H$, and then pull back to $Ult(V, H)$ via j'. □

Generic correctness and stationary tower correctness follows from Lemma 3. So $B_0 \cap V$, $C_0 \cap V$ are in $\text{Hom}_{<\lambda}^V$ from tree production lemma [2], therefore Wadge compatible by $\text{Hom}_{<\lambda}^V$ determinacy. But according to (1) of Lemma 6 with $K = \emptyset$, they are Wadge incompatible using continuous functions coded in $\mathbb{R}^V$. Contradiction.

So (1) of the theorem is proved.

To show (2), we again look for a least counterexample in a certain sense and show the witness is actually in $\mathrm{Hom}^V_{<\lambda}$.

Let $\theta^{\mathcal{A}_G}$ be the height of Wadge degrees of $\mathcal{A}_G$. For any $B \in \mathcal{A}_G$ of Wadge rank β in $\mathcal{A}_G$, denote $\theta_\beta = \theta^{L(B,\mathbb{R}^*_G)}$, and $\kappa_\beta =$ the largest Suslin cardinal of $L(B, \mathbb{R}^*_G)$. Since "κ is a Suslin cardinal" is absolute among models N of AD containing the same reals such that $\kappa < \theta^N$, we have $\beta < \beta' < \theta \to \kappa_\beta \leq \kappa_{\beta'}$ and $\kappa_\beta < \kappa_{\beta'} \to \theta_\beta \leq \kappa_{\beta'}$.

So there are two cases, either κ_β's are cofinal in θ, or $\kappa_\beta = \kappa$ is fixed for arbitrary large $\kappa < \theta$.

In the first case, all sets in $\mathcal{A}_G$ are Suslin-co-Suslin. So $\mathcal{A}_G = \mathrm{Hom}^*_G$. But $L(\mathrm{Hom}^*_G, \mathbb{R}^*_G) \models \mathrm{AD}^+$ from [2].

In the second case, Let Γ be a lightface pointclass for $S(\kappa)$. Let T be the tree on $\omega\times\omega\times\omega\times\kappa$ for the scale on the universal Γ set $U \subset \omega\times\mathbb{R}\times\mathbb{R}$. For $\beta > \kappa$, let $T_\beta = \prod_{\mathcal{D}} T/\mu_M$, computed from functions in $L(B, \mathbb{R}^*_G)$, $w(B) = \beta$. Note that when $\beta < \beta'$, there is an embedding of T_β into $T_{\beta'}$, induced by inclusion. So the wellorder $<^{T_\beta}_{x_g}$ as certified by T_β is an initial segment of $<^{T_{\beta'}}_{x_g}$ as certified by $T_{\beta'}$.

Let $T = \oplus_{\beta<\theta^{\mathcal{A}_G}} T_\beta$ be the amalgamation tree on $\omega\times\omega\times\omega\times\kappa\times\theta^{\mathcal{A}_G}$. We may assume $T \in V$. We let $<^T_{x_g}$ be the wellorder of piecing together all the $<^{T_\beta}_{x_g}$'s, and say that $<^T_{x_g}$ is the wellorder certified by T. The precise statement is:

y codes an $<^T_{x_g}$-initial segment of reals in $L[T, x_g]$ iff there are β, w such that $(k, x_g, y, w, \beta) \in [T]$.

What we want is the following. The rest follows as in [2].

Lemma 7. *Let ϕ be a sentence in the language of set theory with an additional predicate. Suppose $L(\mathcal{A}_G, \mathbb{R}^*_G) \models$ "$\exists B(HC, \in B) \models \phi$". Then there is $B \in \mathrm{Hom}^V_{<\lambda}[G\restriction\alpha]$, some $\alpha < \lambda$, such that $(HC^{V[G\restriction\alpha]}, \in B) \models \phi$.*

Proof. Suppose g is Sacks generic over $L(\mathcal{A}_G, \mathbb{R}^*_G)$. Then

$$L(\mathcal{A}_G, \mathbb{R}^*_G)[g] \models \exists\beta,\gamma\exists B \subset \mathbb{R}, B \in L_\gamma(\mathcal{A}^T_{x_g,<\beta}, \mathbb{R}^*_G)$$
$$(HC^*_G, \in, B) \models \phi.$$

Here

$$\begin{aligned}\mathcal{A}^T_{x_g,<\beta} =&\{E \subset \mathbb{R}^*_G : E \in L(T_\xi, \mathbb{R}^*_G) \text{ for some } \xi \wedge \\ &E \text{ is coded by a real } z \text{ relative to } x_g, \mathbb{R}^*_G \text{ such that } z \text{ appears in} \\ &\text{the wellorder certified by } T \text{ in the } \beta' \text{ position}, \beta' < \beta\}.\end{aligned}$$

The reason is $\mathcal{A}^T_{x_g,<\theta'} = \mathcal{A}_G$ for some θ', which follows from

Claim 1. *Let z be the βth real in $<^T_{x_g}$. Suppose that z codes $E \subset \mathbb{R}^*_G$ relative to x_g and $\mathbb{R}^*_G$. Then whenever $\xi < \eta$ are such that $\beta < {\omega_1}^{L[T_\xi, x_g]}$, $E \in L(T_\xi, \mathbb{R}^*_G)$ iff $E \in L(T_\eta, \mathbb{R}^*_G)$.*

Proof of the claim is similar to lemma 3(iii), so we omit it.

We continue proving lemma 7. let β_0, γ_0 be lexicographically least ordinal witnesses to the fact about $L(\mathcal{A}_G, \mathbb{R}^*_G)[g]$ in the beginning of proof. Fix $z_0 \in \mathbb{R}^*_G$ and $C_0 \in \mathcal{A}^T_{x_g,<\beta}$ so that some ϕ-witness B is ordinal definable from $\{z_0, C_0\}$. The least sequence of ordinals from which one can define a ϕ-witness from $\{z_0, C_0\}$ over $L_{\gamma_0}(\mathcal{A}^T_{x_g,<\beta_0}, \mathbb{R}^*_G)$ is definable from $\{z, C\}$, and so we may assume that B is definable without parameters. We also assume that C_0 is coded by the ξ_0th real certified by T relative to $x_g, \mathbb{R}^*_G$, and ξ_0 is least which codes $C \in \mathcal{A}^T_{x_g,<\beta_0}$ such that there is a ϕ-witness definable from $\{z_0, C\}$. Let χ define this ϕ-witness B_0:

$$x \in B_0 \leftrightarrow L_{\gamma_0}(\mathcal{A}^T_{x_g,<\beta_0}, \mathbb{R}^*_G) \models \chi(z_0, C_0, x).$$

let $\psi'(\beta_0, \gamma_0, z_0, \xi_0, C_0, B_0, T, x_g, \mathbb{R}^*_G)$ be conjunction of the following formulas:

(i) β_0, γ_0 is lexicographically least β, γ so that there is a ϕ-witness $B \in L_\gamma(\mathcal{A}^T_{x_g,<\beta}, \mathbb{R}^*_G)$

(ii) ξ_0 is the least ξ so that letting $C \subset \mathbb{R}^*_G$ be coded by the ξth real in $<^T_{x_g}$ relative to x_g and $\mathbb{R}^*_G$, we have $C \in \mathcal{A}^T_{x_g,<\beta_0}$, and there is a ϕ-witness definable from $\{z_0, C\}$ over $L_{\gamma_0}(\mathcal{A}^T_{x_g,<\beta_0}, \mathbb{R}^*_G)$

(iii) C_0 is coded by the ξ_0th real in $<^T_{x_g}$ relative to x_g and $\mathbb{R}^*_G$,

(iv) $B_0 = \{x \in \mathbb{R} : L_{\gamma_0}(\mathcal{A}^T_{x_g,<\beta_0}, \mathbb{R}) \models \chi(z_0, C_0, x)\}$, and is a ϕ-witness.

Then $L(\mathcal{A}_G, \mathbb{R}^*_G)[g] \models$ "$\psi'(\beta_0, \gamma_0, z_0, \xi_0, C_0, B_0, T, x_g, \mathbb{R}^*_G)$ holds and $(\beta_0, \gamma_0, \xi_0, C_0, B_0)$ is the unique such tuple". Let p be a condition forcing $\psi'(\check{\beta}_0, \check{\gamma}_0, \check{z}_0, \check{\xi}_0, \check{C}_0, \check{B}_0, \check{T}, \dot{x}_g, \check{\mathbb{R}}^*_G)$ over $L(\mathcal{A}_G, \mathbb{R}^*_G)$. We may assume that $p, z_0 \in V$. Let $\phi_3(v, p, z_0, x)$ be the formula

> There are $\beta', \gamma', \xi', C', B'$ such that $p \Vdash_{\mathbb{P}} \psi'(\check{\beta}', \check{\gamma}', \check{z}_0, \check{\xi}, \check{C}', \check{B}', \check{v}, \dot{x}_g, \check{\mathbb{R}})$ and $x \in B'$.

We want to show that $\phi_3(T, p, z_0, x)$ defines $B \in \mathrm{Hom}^V_{<\lambda}$ over V.

Lemma 8. *Let G be $\mathbb{Q}_{<\lambda}$ generic over V. G then induces generic ultrapower embeddings $j_\delta : V \to M_\delta = Ult(V, G \cap V_\delta)$ and their direct limit $j^* : V \to M = \mathrm{dirlim}_{\delta<\lambda} M_\delta$. Suppose that $\mathbb{R} \cap M = \mathbb{R}^*_G$ and that $\theta^{\mathcal{A}_G}, \gamma_0, \xi_0 \in \mathrm{wfp}(M)$. Suppose that h is Sacks generic over both $V(\mathbb{R}^*_G)$ and M such that $p \in h$. Then $\psi'(\beta_0, \gamma_0, z_0, \xi_0, C_0, B_0, j^*(T), x_h, \mathbb{R}^*_G)$ holds in $M[x_h]$, and $(\beta_0, \gamma_0, \xi_0, C_0, B_0)$ is the unique such tuple.*

Proof. Firstly note that $j^*(T)$ certifies a wellorder $<^{j^*(T)}_{x_h}$ in M (may not be a true wellorder). The reason is that, the tree S_T of attempting $(\beta, \beta', x, y_1, y_2, w_1, w_2)$ with $(k, x, y_1, w_1, \beta), (k, x, y_2, w_2, \beta') \in [T]$ and either y_1 does not code a wellorder or y_1, y_2 are incompatible, is wellfounded. Hence $S_{j^*(T)}$ is wellfounded in M.

Secondly, $<^{j^*(T)}_{x_h}$ has a length $\beta_0 + 1$ truly wellordered initial segment, since M has $\beta_0 + 1$ in its wellfounded part. That initial segment of $<^{j^*(T)}_{x_h}$ is also an initial segment of $<^T_{x_h}$, since letting $\xi < \theta^{\mathcal{A}_G}$ be large enough that $\beta_0 < {\omega_1}^{L[T_\xi, x_h]}$, we know from Lemma 3(ii) that $<^{j^*(T)\xi}_{x_h}$ has an length $\beta_0 + 1$ initial segment which is also an initial segment of $<^{T_\xi}_{x_h}$.

Finally we are able to show that $\mathcal{A}^{j^*(T)}_{x_h, <\beta} = \mathcal{A}^T_{x_h, <\beta}$ for small β's. This follows from

Claim 2. *Let z be the βth real in $<^T_{x_g}$, where $\beta \leq \beta_0$. Suppose that z codes $E \subset \mathbb{R}^*_G$ relative to x_g and $\mathbb{R}^*_G$. Then whenever $\xi < \eta$ are such that $\beta < {\omega_1}^{L[(j(T))_\xi, x_g]}$, $E \in L(T_\xi, \mathbb{R}^*_G)$ iff $E \in L((j^*(T))_\xi, \mathbb{R}^*_G)$ iff $E \in L((j^*(T))_\eta, \mathbb{R}^*_G)$.*

Proof of Claim 2 is similar to that of Lemma 3(iii), so we omit it. So part (i) of $\psi'(\beta_0, \gamma_0, z_0, \xi_0, C_0, B_0, j^*(T), x_h, \mathbb{R}^*_G)$ is satisfied in $M[x_h]$. Parts (ii)-(iv) are also satisfied because of $\mathcal{A}^{j^*(T)}_{x_h, <\beta_0} = \mathcal{A}^T_{x_h, <\beta_0}$ and $L_{\gamma_0}(\mathcal{A}^{j^*(T)}_{x_h, <\beta_0}, \mathbb{R}^*_G) = L_{\gamma_0}(\mathcal{A}^T_{x_h, <\beta_0}, \mathbb{R}^*_G)$. Uniqueness of $(\beta_0, \gamma_0, \xi_0, C_0, B_0)$ is then immediate from definition of ψ'. This completes the proof of Lemma 8. $\square$

Corollary 9. *Let M, j^* be as in Lemma 8. Then for $x \in \mathbb{R}^*_G$,*

$$x \in B_0 \leftrightarrow M \models \phi_3(j^*(T), p, z_0, x).$$

Generic correctness and stationary tower correctness follows from Corollary 9. So $\{x \in \mathbb{R}^V : \phi_3(T, p, z_0, x)\}$ is a ϕ-witness in V. This completes the proof of Lemma 7, and hence Theorem 1.

□

References

[1] J. Steel, Notes on AD^+, handwritten unpublished notes.

[2] J. Steel, The derived model theorem, available at www.math.berkeley.edu/∼steel.