ON THE REALS WHICH CANNOT BE RANDOM

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ABSTRACT. We investigate which reals can never be L-random. That is to give a description of the reals which are always belong to some $L[\lambda]$-null set for any continuous measure λ. Among other things, we prove that NCR_L is an L-cofinal subset of Q_3 under $ZFC + PD$.

1. Introduction

This paper is inspired by the work of Reimann and Slaman [11] and [12].

A real x is called *never continuous random* (NCR_1) if there is no continuous measure λ so that x is Martin-Löf random with respect to λ. In both papers, a fairly clear description of NCR_1 was given. For example, they proved that NCR_1 is a subset of the collection of hyperarithmetic reals and contains all the reals which belong to some countable Π^0_1-set.

Martin-Löf randomness may be a "real" randomness notion from a *computability theorist* point view. In this paper, we investigate L-randomness, the randomness relative to constructibility, which may be viewed as an "actual" randomness notion.

The L-randomness notion was introduced by Solovay in his celebrated paper [16]. A real r is L-random if it does not belong to any Borel null set which has a Borel code in L. We may generalize this notion to any continuous measure λ and introduce $L[\lambda]$-randomness. Then a notion of *never L-continuous random* (NCR_L) can be naturally defined. The target of this paper is to give a description of NCR_L.

It turns out that NCR_L becomes interesting only under certain large cardinal assumptions. If people think of that Π^1_2-ness and Σ^1_3 "correspond" to Π^0_1-ness and Σ^1_1-ness respectively under PD, then many results in [11] and [12] can be lifted.

We organize the paper as follows: In Section 2, we give non self-contained preliminaries for the further reading. In Section 3, we investigate NCR_L under certain fairly weak set theory assumptions (not stronger than the existence of an inaccessible cardinal). In Section 4, we give a description of NCR_L under PD.

2. Preliminaries

Since a lot of facts from set theory, recursion theory and algorithmic randomness theory are needed, we feel that it is unlikely to give a self-contained preliminary. We mostly follow standard terminology and notations from the standard references like [5], [15] and [3] to make the paper accessible to readers.

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We identify an open set in 2^ω with its representation, a subset of $2^{<\omega}$. For a finite string $\sigma \in 2^{<\omega}$, we use $[\sigma]$ to denote the basic open set $\{x \in 2^\omega \mid x \succ \sigma\}$.

First note that if λ is a finite Borel measure, then it is uniquely determined by a measure $\widetilde{\lambda}$ over open sets. Throughout this paper, we only consider Borel measures. So they all have standard representations.

Definition 2.1. *For any measure λ over 2^ω, we use $\hat{\lambda} \in \mathbb{Q}^2 \times 2^{<\omega}$ to denote its standard representation $\{(p, q, \sigma) \mid \lambda(\sigma) \in [p, q]\}$.*

From now on, *we identify a Borel measure with its representation.*

Definition 2.2. *A probability measure λ over 2^ω is a Borel measure so that*
1. *$\lambda(2^\omega) = 1$; and*
2. *For any $\sigma \in 2^{<\omega}$, $\lambda([\sigma]) = \lambda([\sigma^\frown 0]) + \lambda([\sigma^\frown 1])$.*

Definition 2.3. *A* continuous *measure λ over 2^ω is a probability Borel measure so that for any real x, $\lambda(\{x\}) = 0$.*

Note that a probability measure λ is continuous if and only if for any n, there is some m so that for any $\sigma \in 2^m$, $\lambda([\sigma]) \leq 2^{-n}$. So we have the following result.

Lemma 2.4. *The set $\{\hat{\lambda} \mid \hat{\lambda}$ represents a continuous measure$\}$ is Δ^1_1.*

Definition 2.5. *For any real x and measure λ, a real r is $L[\lambda \oplus x]$-λ-*random *if for any λ-null Borel set A which has a Borel code in $L[\hat{\lambda} \oplus x]$, $x \notin A$.*

If $x \in L[\hat{\lambda}]$ and r is $L[\lambda \oplus x]$-λ-random, we simply say that r is $L[\lambda]$-random. Further more, if λ is Lesbegue measure and r is $L[\lambda]$-random, then we simply say that r is L-random.

Definition 2.6.

$$NCR_L = \{x \mid \textit{For any continuous measure } \lambda, x \textit{ is not } L[\lambda]\textit{-random.}\}.$$

Lemma 2.7. *NCR_L is Π^1_3.*

Proof. $x \in NCR_L$ if and only if for any $\hat{\lambda}$, if $\hat{\lambda}$ represents a continuous measure, then there is a Borel set A having a Borel code in $L[\hat{\lambda}]$ so that $\lambda(A) = 0$ and $x \in A$. By Lemma 2.4 and some well known descriptive set theory result (see [10] or [2]), NCR_L is Π^1_3. □

The following proposition is routine.

Proposition 2.8. *If λ is a continuous measure, then r is $L[\lambda \oplus x]$-λ-random if and only if for any $\mathbf{\Pi}^0_2$-λ-null set A having a Borel code in $L[\hat{\lambda} \oplus x]$, $x \notin A$.*

Fix a real x and continuous measure λ, let $\mathbb{P}_{\lambda,x} = (\mathbf{P}_{\lambda,x}, \leq)$ be a λ-x-Solovay forcing so that
1. $P \in \mathbf{P}_{\lambda,x}$ if and only if P is a closed non-λ-null set in $L[\hat{\lambda} \oplus x]$; and
2. For two conditions P_0 and P_1, $P_0 \subseteq P_1$ if and only if $P_0 \leq P_1$.

If $x \in L[\hat{\lambda}]$, then simply use $\mathbb{P}_\lambda$ to denote $\mathbb{P}_{\lambda,\hat{\lambda}}$.

$\mathbb{P}_{\lambda,x}$ has almost all the properties of classical Solovay forcing. For example, it is c.c.c and has the homogeneity property.

The following proposition is obvious.

Proposition 2.9. *Fix a real x and continuous measure λ, a real r is $L[\lambda \oplus x]$-λ-random if and only if r is a $\mathbb{P}_{\lambda,x}$-generic real over $L[\hat{\lambda} \oplus x]$.*

3. Basic results

In this section, we investigate NCR_L under weak set theoretic hypotheses (not stronger than the existence of an inaccessible cardinal).

The following result can be viewed as a set theoretical version of Demuth's theorem (see [9]).

Theorem 3.1. *For any real x, continuous measure λ, $L[\lambda \oplus x]$-λ-random real r, if $z \in L[\hat{\lambda} \oplus x \oplus r] \setminus L[\hat{\lambda} \oplus x]$, then z is $L[\lambda \oplus x \oplus \rho]$-$\rho$-random with respect to some continuous measure $\rho \in L[\hat{\lambda} \oplus x]$. In particular, if r is L-random and $z \in L[r] \setminus L$, then z is $L[\rho]$-random with respect to some continuous measure $\rho \in L[r]$.*

Proof. Suppose that r is $L[\lambda \oplus x]$-λ-random and $z \in L[\hat{\lambda} \oplus x \oplus r] \setminus L[x \oplus \hat{\lambda}]$. Then there is a condition $P \in \mathbb{P}_{\lambda,x}$ such that $P \Vdash \dot{z} \in \dot{2^\omega}$ and $r \in P$. Since $\mathbb{P}_{\lambda,x}$ is c.c.c, there is a sequence of conditions $\{P_n^i \mid i, n \in \omega\} \in L$ below P so that

- $\forall i \forall n \exists j_i (P_n^i \Vdash \dot{x}(\check{n}) = \check{j}_i)$; and
- For all n, $\{P_n^i\}_{i\in\omega}$ is a maximal antichain below P.

Note that for each n, there is only one k_n such that $r \in [P]_n^{k_n}$. Then there is a function $f \in L[\hat{\lambda} \oplus x]$ such that for any $i \neq k_n$, $r \upharpoonright f(\langle i, n\rangle) \notin P_n^i$. Since random forcing is dominated (i.e. $2^{<\omega} \Vdash \forall f \exists g \in L[\check{\lambda} \oplus \check{x}] \forall n (f(n) \leq g(n))$, there is a function $g \in L[\hat{\lambda} \oplus x]$ such that g dominates f. Hence we may code the sequence $\{P_n^i \mid i, n \in \omega\}$ and the relation $P_n^i \Vdash \dot{z}(\check{n}) = \check{j}_i$ into a single real $t = \{\langle i, n, \sigma, j_i\rangle \mid \sigma \in P_n^i \wedge P_n^i \Vdash \dot{z}(\check{n}) = \check{j}_i\} \in L[\hat{\lambda} \oplus x]$. Now for each n, we $r \oplus t \oplus g$-recursively find an i such that $r \upharpoonright g(\langle i, n\rangle) \in P_n^i$. Then $i = k_n$ as above. Then there is a unique j_{k_n} such that $\langle k_n, n, r \upharpoonright g(\langle k_n, n\rangle), j_{k_n}\rangle \in t$. So $z(n) = j_{k_n}$. In other words,

$$z = \Psi^{r \oplus t \oplus g}$$

for some Turing functional Ψ. Again since random forcing only adds dominated functions, there is a function $h_0 \in L[\hat{\lambda} \oplus x]$ that dominates the use function of $\Psi^{r \oplus t \oplus g}$.

By the dominated property again and the fact that $z \notin L[\hat{\lambda} \oplus x]$, we may assume that there is a non-decreasing function $h_1 \in \omega^\omega \cap L[\lambda \oplus x]$ so that

- $\lim_{n\to\omega} h_1(n) = \infty$; and
- $h_1(0) = 0$; and
- $\forall n (\lambda(\{y \mid z \upharpoonright n = \Psi^{y \oplus t \oplus g} \upharpoonright n\}) \leq 2^{-h_1(n)})$.

For any $\tau \in 2^{<\omega}$, let

$$C(\tau) = \{\sigma \mid \sigma \in 2^{h_0(|\tau|)} \wedge \Psi^{\sigma \oplus t \upharpoonright h_0(|\tau|) \oplus g \upharpoonright h_0(|\tau|)}[h_0(|\tau|)] \succeq \tau\}.$$

Inductively define $\rho \in L[\hat{\lambda} \oplus x]$ as follows:

$$\rho(\emptyset) = 1, \text{ and}$$

$$\rho(\tau^\frown i) = \begin{cases} \lambda(\bigcup_{\sigma \in C(\tau^\frown i)}[\sigma]), & \forall \tau' \preceq \tau(\rho(\tau') \leq 2^{-h_1(|\tau'|)}); \\ \frac{\rho(\tau)}{2}, & \text{Otherwise.} \end{cases}$$

Note that for any τ,

$$C(\tau) = C(\tau^\frown 0) \cup C(\tau^\frown 1), \text{ and } C(\tau^\frown 0) \cap C(\tau^\frown 1) = \emptyset.$$

Since λ is a probability measure, for any τ with the property that $\forall \tau' \preceq \tau(\rho(\tau') \leq 2^{-h_1(|\tau'|)})$, it must be that $\rho(\tau) = \rho(\tau^\frown 0) + \rho(\tau^\frown 1)$. Then one may easily check that ρ is induces a probability measure. Since the limit of h_1 is infinite and λ is continuous, ρ must be continuous.

Now suppose that $\{U_n\}_{n\in\omega}$ in $L[\hat{\lambda} \oplus x]$ is a descending sequence of open sets so that $z \in \bigcap_{n\in\omega} U_n$ and $\rho(\bigcap_{n\in\omega} U_n) = 0$. Define a sequence of open sets $\{\hat{U}_n\}_{n\in\omega}$ so that for any n,

$$\tau \in \hat{U}_n \text{ iff } \forall \tau' \preceq \tau(\rho(\tau') \leq 2^{-h_1(|\tau'|)}).$$

Then $\forall n(\hat{U}_n \subseteq U_n)$ and $z \in \bigcap_{n\in\omega} \hat{U}_n$.

Now for every n, let $V_n = \{\sigma \mid \exists \tau \in \hat{U}_n(\sigma \in C(\tau))\}$. Note that for every n,

$$\lambda(V_n) = \sum_{\exists \tau \in \hat{U}_n(\sigma \in C(\tau))} \lambda([\sigma]) = \sum_{\tau \in \hat{U}_n} \rho(\tau) = \rho(\hat{U}_n) \leq \rho(U_n).$$

Moreover since $z \in \bigcap_{n\in\omega} \hat{U}_n$, by the definition of V_n, we have that $r \in V_n$ for every n.

So $\{V_n\}_{n\in\omega}$ in $L[\hat{\lambda} \oplus x]$ is a descending sequence of open sets so that $r \in \bigcap_{n\in\omega} V_n$ and $\rho(\bigcap_{n\in\omega} V_n) = 0$. Then r is not $L[\hat{\lambda} \oplus x]$-$\lambda$-random, a contradiction.

Hence z must be $L[\lambda \oplus x]$-ρ-random and so $L[\lambda \oplus x \oplus \rho]$-$\rho$-random. □

We use $x \equiv_L y$ to denote $x \in L[y]$ and $y \in L[x]$. Then immediately, we have the following result.

Corollary 3.2. *If $x \in NCR_L$ and $x \equiv_L y$, then $y \in NCR_L$.*

Obviously if $2^\omega \subseteq L$, then $NCR_L = 2^\omega$.

Proposition 3.3. (1) *If $NCR_L \neq 2^\omega$, then NCR_L is not Π^1_2.*
(2) *If $V = L[r]$ for some L-random real r, then NCR_L is Σ^1_2.*
(3) *If $(\aleph_1)^{L[x]}$ is countable for any real x, then NCR_L is thin; and NCR_L is Σ^1_2 if and only if $NCR_L \subseteq L$.*

Proof. (1). If $NCR_L \neq 2^\omega$ and NCR_L is Π^1_2, then $2^\omega \setminus NCR_L$ is a nonempty Σ^1_2-set and so must a contain a real in L, which is a contradiction to the Shoenfield's absolutness.

(2). If $V = L[r]$ for some L-random real r, then by Theorem 3.1, $NCR_L = 2^\omega \cap L$ and so must be Σ^1_2.

(3). Suppose that for any real x, $(\aleph_1)^{L[x]}$ is countable and NCR_L is not thin. Then there is a perfect tree $T \subseteq 2^{<\omega}$ so that $[T] \subseteq NCR_L$. Define a continuous measure ρ "focusing on T" in $L[T]$ as follows:

$$\rho(\emptyset) = 1, \text{ and}$$

$$\rho(\sigma^\frown i)=\begin{cases}\frac{\rho(\sigma)}{2}, & \sigma^\frown i\in T\wedge \sigma^\frown(1-i)\in T;\\ \rho(\sigma), & \sigma^\frown i\in T\wedge \sigma^\frown(1-i)\notin T;\\ 0, & \text{Otherwise};\end{cases}$$

Then $\rho([T])=1$. Since $2^\omega\cap L[T]$ is countable, there must be some $L[T]$-ρ-random real $r\in[T]$, a contradiction.

Now if NCR_L is Σ^1_2, then $NCR_L\subseteq L$ since $2^\omega\cap L$ is the largest Σ^1_2-thin set. If $NCR_L\subseteq L$, then $NCR_L=2^\omega\cap L$ and so must be Σ^1_2. □

The following fact gives a plenty of examples of NCR_L.

Proposition 3.4. *Suppose that for any real x, $(\aleph_1)^{L[x]}$ is countable. If $A\subseteq 2^\omega$ is a Π^1_2-thin set, then $A\subseteq NCR_L$.*

Proof. Suppose that φ is a Π^1_2-formula and x is a real so that $\varphi(x)$ and the set $\{y\mid \varphi(y)\}$ is countable. Suppose that there is a continuous measure ρ so that x is $L[\rho]$-random. Note that $L[\rho\oplus x]\models\varphi(x)$ by the Shoenfield absoluteness. Then $p\Vdash\varphi(\dot{x})$ for some condition $p\in\mathbf{P}_\rho$. Then by the homogeneity of random forcings, for any $L[\rho]$-random real $y\in p$, $L[\rho\oplus y]\models\varphi(y)$. By Shoenfield absoluteness again, $\varphi(y)$ is true. Since there are countably many reals in $L[\rho]$, there must be ρ-conull many $L[\rho]$-random reals. So $\{y\mid\varphi(y)\}$ has a perfect subset, a contradiction. □

We don't know whether the assumption of Proposition 3.4 can be weakened to a non-large cardinal one. But note that if r is an L-random, then $\{x\mid x\in L[r]\text{ is }L\text{-random}\}$ is a Π^1_2 thin set (see Theorem 3.2.17 in [1]).

We also remark that the union of all Π^1_2-thin sets is a proper subset of NCR_L. Actually Friedman proved [4] that there is a Δ^1_3 real $x\in L$ which does not belong to any Π^1_2-countable set.

4. Under PD

Throughout this section, we assume that ZFC+projective determinacy, PD.

By (3) of Proposition 3.3, NCR_L is a Π^1_3 countable set.

We need the following theorem which can be found in [5] (Exercise 18.6).

Theorem 4.1 (Kunen)**.** *If κ is weakly compact and $|(\kappa^+)^L|=\kappa$, then $0^\sharp$ exists.*

Definition 4.2. *Let $j:2^\omega\to Ord$ be a function so that $\forall x(j(x)=(\kappa^+)^{L[x]})$ where $\kappa=\aleph_1$*[1]*.*

Lemma 4.3 (Simpson [14])**.** *The function j has the following property:*

(1) *$x\in L[y]\to j(x)\le j(y)$; and*
(2) *$x\in L[y]\to(j(x)<j(y)\leftrightarrow x^\sharp\in L[y])$.*

(1) is obvious. To see (2), for any real x, note that κ is weakly compact in $L[x]$. So if $x\le_L y$ and $j(x)<j(y)$, then $L[y]\models|(\kappa^+)^{L[x]}|=\kappa$. Then by Theorem 4.1 relative to x, $L[y]\models x^\sharp$ exists. So $x^\sharp\in L[y]$. Another direction of (2) is obvious.

Proposition 4.4. $0^\sharp\in NCR_L$.

[1]The function j was introduced in [14]. We use κ to denote $\aleph_1$ in V to avoid any confusion.

Proof. This follows from Proposition 3.4 since $0^\sharp$ is a Π^1_2-singleton.

We give alternative proof that is forcing-free. By Theorem 4.1 again, for any continuous measure $\lambda \not\geq_L 0^\sharp$, $j(\lambda) < j(0^\sharp) \leq j(\lambda \oplus 0^\sharp)$. So by (2) above, $\lambda \oplus 0^\sharp \geq_L \lambda^\sharp$. Let $\kappa = \aleph_1$, then $(\kappa^+)^{L[\lambda]} < (\kappa^+)^{L[\lambda \oplus 0^\sharp]}$. Since random forcing does not collapse cardinals, $0^\sharp$ cannot be $L[\lambda]$-random.

□

The general form of Proposition 4.4 will be proved in Lemma 4.21 and Theorem 4.22, using the covering property for the core model below one Woodin cardinal.

So by Proposition 4.4, and (1) and (3) of Proposition 3.3, we have the following corollary.

Corollary 4.5. *NCR_L is neither Π^1_2 nor Σ^1_2.*

However, NCR_L is not closed under Δ^1_3-equivalence relations. For example, there is an L-random real r Turing below $0^\sharp$. Then the real r must be Δ^1_3.

Let $\mathcal{C}_3$ be the largest countable Π^1_3-set.

The existence of $\mathcal{C}_3$ was proved by Kechris [7]. By the discussion above, $NCR_L \subset \mathcal{C}_3$. We will show that NCR_L lives inside the "bottom" of $\mathcal{C}_3$.

Definition 4.6. *$Q_3 = \{x \mid \exists \alpha < \omega_1 \forall z(\omega_1^z > \alpha \to x \leq_{\Delta^1_3} z)\}$, where ω_1^z is the least non-z-recursive ordinal.*

In [8], it was proved that Q_3 is a Π^1_3 countable set which is downward closed under $\leq_{\Delta^1_3}$. One may also relativize the definition of Q_3 to any real x and obtain $Q_3(x)$. Then it induces a reduction $y \leq_{Q_3} x$ iff $y \in Q_3(x)$. Just like in the higher recursion theory, any two reals in $\mathcal{C}_3$ are Q_3-comparable, and $\mathcal{C}_3$ is closed under Q_3-equivalence relation, and every real in the least Q_3-degree above $\mathbf{0}$ is a Π^1_3-singleton.

We choose a representative $y_{0,3}$ as a Q_3-complete real so that every nonempty Σ^1_3 set contains a real recursive in $y_{0,3}$. Note that $y_{0,3}$ is far more complex than the Π^1_3-complete real which actually belongs to Q_3 (see [8]).

Lemma 4.7. *For any real x, there is a real $y \geq_T x$ such that there is a continuous measure $\rho \leq_T y$ so that y is $L[\rho]$-random.*

Proof. For any x, let r be $L[x]$-random.

Let $\rho \leq_T x$ be a continuous measure so that

$$\rho(\emptyset) = 1, \text{ and}$$

$$\rho(\sigma^\frown i) = \begin{cases} \frac{\rho(\sigma)}{2}, & |\sigma| \text{ is odd}, \\ \rho(\sigma), & |\sigma| \text{ is even} \wedge i = x(\frac{|\sigma|}{2}), \\ 0, & \text{Otherwise}. \end{cases}$$

Since r is $L[x]$-random, it is not difficult to see that $y = x \oplus r$ is $L[\rho]$-random. □

Definition 4.8. *Let NCR_L^T be the set of reals r so that there is no continuous measure $\rho \leq_T r$ so that r is $L[\rho]$-random.*

Then NCR_L^T is a Σ^1_2-set and so $NCR_L^T \neq NCR_L$. By Lemma 4.7, $2^\omega \setminus NCR_L^T$ has cofinally many L-degrees and so contains an upper cone of L-degrees.

Proposition 4.9. *Every Sacks generic real g over L belongs to NCR_L^T.*

Proof. For a contradiction, suppose that g is a Sacks generic real over L and g is $L[\rho]$-random with respect to some continuous measure $\rho \in L[r]$. Then $\rho \in L$.

It is well known (see [1]) that for any function $f \in L[r] \cap \omega^\omega$, there is a function $t \in L \cap (\mathcal{P}_{<\omega}(\omega))^\omega$, where $\mathcal{P}_{<\omega}$ is the collection of finite subsets of ω, so that for any n,

- $f(n) \in t(n)$; and
- $|t(n)| < n$.

Since ρ is a continuous measure, there is a function $\hat{h} \in L[g] \cap \omega^\omega$ so that for any n, $\rho([x \upharpoonright \hat{h}(n)]) < 2^{-n}$. By the dominated property of Sacks forcing, there is a function $h \in L \cap \omega^\omega$ that dominates $\hat{h}$.

Then let $f \in L[g]$ be a function so that $f(n) = g \upharpoonright h(n)$. Then let t be as above.

Now for each n, let $V_n = \{\sigma \in 2^{h(n)} \cap t(n) \mid \rho([\sigma]) < 2^{-n}\}$. Then $\{V_n\}_{n\in\omega}$ is a sequence in L so that

- $\forall n(\rho(V_n) \leq n \cdot 2^{-n})$; and
- $g \in \bigcap_{n\in\omega} V_n$.

So g is not $L[\rho]$-random, a contradiction. □

Corollary 4.10. *NCR_L^T contains a perfect subset.*

Actually by the proof above, for any real $x \in P_2$, $NCR_L^T \cap \{y \mid x \in L[y]\}$ contains a perfect subset.

Let
$$D = \{y_0 \mid \forall y(y \geq_T y_0 \to y \notin NCR_L^T)\}.$$
By PD and Lemma 4.7, D is a nonempty Π^1_2-set and so the Q_3-complete real $y_{0,3} \in D$.

By relativizing the discussion above, we have the following lemma.

Lemma 4.11. *For any real z and real $x \geq_T y^z_{0,3}$, where $y^z_{0,3}$ is the Q_3-jump relative to z, there is a continuous measure $\rho \leq_T x \oplus z$ so that x is $L[z \oplus \rho]$-ρ-random.*

We need the following Posner-Robinson Theorem.

Theorem 4.12 (Woodin)**.** *If $x \notin Q_3$, then there is a real z so that $x \oplus z \geq_T y^z_{0,3}$.*

Woodin's proof remains unpublished, though it is confirmed by him. Hopefully we may figure it out in the near future.

Lemma 4.13. *For any $x \notin Q_3$, there is a real z so that $x \oplus z$ is $L[z \oplus \rho]$-random with respect to some continuous measure $\rho \leq_T x \oplus z$. Further more, x must be $L[\rho_0]$-random with respect to some continuous measure ρ_0.*

Proof. Suppose that $x \notin Q_3$, then by Posner-Robinson Theorem 4.12, there is a real z so that $x \oplus z \geq_T y^z_{0,3}$. By Lemma 4.11, $x \oplus z$ is $L[z \oplus \rho]$-ρ-random with respect to some measure $\rho \leq_T x \oplus z$.

Obviously $x \not\leq_L \rho \oplus z$. Then by Theorem 3.1, x is $L[z \oplus \rho \oplus \rho_0]$-$\rho_0$-random with respect to some continuous measure $\hat{\rho}_0 \in L[z \oplus \hat{\rho} \oplus \hat{\rho}_0]$. In particular, x is $L[\rho_0]$-random. □

In summary, we have the following theorem.

Theorem 4.14. $NCR_L \subseteq Q_3$.

Now we want to know how "large" is NCR_L.

Proposition 4.15. *For every Δ^1_3-real x, there is a Π^1_2-singleton z so that $x \leq_T z$.*

Proof. If x is Δ^1_3, then $\{x\}$ is also Δ^1_3. So by 6E.14 in [10], there is a Π^1_2 set A and a recursive function $f : \omega^\omega \to 2^\omega$ so that $f(A) = \{x\}$ and f is $1-1$ over A. Obviously there is a $z \in \omega^\omega$ so that $A = \{z\}$. So $x \leq_T z$ and z is a Π^1_2-singleton. □

So by Proposition 3.4, $NCR_L \cap \Delta^1_3$ is L-cofinal in the collection of Δ^1_3-reals.

Definition 4.16. $P_2 = \{x \mid \forall y(j(x) \leq j(y) \to x \in L[y])\}$.

Lemma 4.17. (1) $0 \in P_2$.
(2) $x \in P_2 \wedge z \equiv_L x \implies z \in P_2$.
(3) $x \in P_2 \implies x^\sharp \in P_2$.
(4) $P_2 \subseteq NCR_L$.

Proof. (1) is obvious.
If $z \equiv_L x$, then $j(x) = j(z)$. So (2) is true.
If $x \in P_2$ and $j(x^\sharp) \leq j(y)$, then $j(x) < j(y)$. So $x \in L[y]$. By Lemma 4.3, $x^\sharp \leq_L y$. So (3) is proved.
To see (4), suppose that $x \in P_2 \setminus NCR_L$, then x is $L[\lambda]$-random for some continuous measure λ. Since random forcing preserves all cardinals, we have that $j(x) \leq j(\lambda)$. Since $x \in P_2$, we also have that $x \in L[\hat{\lambda}]$, which is a contradiction. □

From this point on, we require basic knowledge of inner model theory in the region of one Woodin cardinal. We shall follow the notations in [20]. Briefly speaking, a premouse is a model of the form $\mathcal{M} = J^{\vec{E}}_\alpha$ with certain fine structural properties and coherency properties, where $\vec{E}$ codes a sequence of extenders. $\rho(\mathcal{M})$ denotes the ultimate projectum of $\mathcal{M}$. $o(\mathcal{M})$ denotes the height of $\mathcal{M}$. $\mathcal{M}|\xi$ denotes the initial segment of $\mathcal{M}$ of height ξ, that is, $J^{\vec{E}}_\alpha|\xi = J^{\vec{E}}_\xi$. $\mathcal{M} \trianglelefteq \mathcal{N}$ means $\mathcal{M}$ is an initial segment of $\mathcal{N}$. A normal iteration tree $\mathcal{T}$ on a premouse $\mathcal{M}$ consists of premice $(\mathcal{M}_\alpha : \alpha < \lambda)$, extenders $(E_\alpha : \alpha < \lambda)$, a tree order T on λ, and a set $D \subseteq \lambda$. Here, E_α is an extender on the $\mathcal{M}_\alpha$-sequence, $\mathcal{M}_{\alpha+1}$ is the fine-structural ultrapower of an initial segment of $\mathcal{M}_{T\text{-pred}(\alpha+1)}$ according to E_α, $\alpha < \beta \to \mathrm{lh}(E_\alpha) < \mathrm{lh}(E_\beta)$, D is the set of dropping points. If $\alpha <^T \beta$ and $D \cap (\alpha, \beta]_T = \emptyset$ then we have an iteration map $i_{\alpha\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$.

If $\mathcal{T}$ is an iteration tree of limit length, in order to continue the iteration, we need to find a cofinal branch through $\mathcal{T}$. The branch choice is at the level of one Woodin cardinal is handled in [20, Theorem 6.10]. We denote by $\mathcal{M}(\mathcal{T})$ the common part of $\mathcal{T}$, $\delta(\mathcal{T})$ the sup of lengths of extenders used in $\mathcal{T}$, as in [20, Definition 6.9]. If b is a cofinal branch through $\mathcal{T}$, then $\mathcal{M}^{\mathcal{T}}_b$ is the (not necessarily wellfounded) direct limit of models along b, and $\mathcal{Q}(b, \mathcal{T})$ is the least initial segment of $\mathcal{M}^{\mathcal{T}}_b$ which either projects across $\delta(\mathcal{T})$ or defines a failure of Woodinness of $\delta(\mathcal{T})$ at the next level[2].

[2]This is slightly different from [20, Definition 6.11]. In our situation, b needs not be wellfounded.

By the proof of [20, Corollary 6.14], if b, c are both cofinal branches through $\mathcal{T}$ and $\mathcal{Q}(b, \mathcal{T}) = \mathcal{Q}(c, \mathcal{T})$, then $b = c$.

If there is no inner model with a Woodin cardinal, the core model K exists. K is defined initially by Steel in [19] as an inner model of V_Ω for a measurable cardinal Ω, and later by Jensen-Steel in [6] from ZFC alone. The iteration strategy for K or any mouse $\mathcal{M}$ is simply as follows: if $\mathcal{T}$ is an iteration tree of limit length, choose the unique branch b through $\mathcal{T}$ such that $Q(b, \mathcal{T})$ exists and $Q(b, \mathcal{T}) = J_\xi[\mathcal{M}(T)]$ for some ordinal ξ. Such an iteration strategy is called the L-guided strategy, in the sense that $\mathcal{Q}(b, \mathcal{T})$ is an initial segment of $L[\mathcal{M}(\mathcal{T})]$. If N is a model of ZFC and "no inner model with a Woodin cardinal", K^N denotes the core model defined in N.

If $\mathcal{M}$ is a sound premouse projecting to ω, its master code is (modulo arithmetic equivalence) the first order theory of $\mathcal{M}$, coded into a real.

We use M_1 to denote the structure introduced by Steel in [17] which is the least inner model containing a Woodin cardinal. A master code in M_1 is a master code of $\mathrm{M}_1|\alpha$ for some α so that $\rho(\mathrm{M}_1|\alpha) = \omega$. Note that every real in M_1 is recursive in a master code in M_1.

The connection between the theory of M_1 and Q_3-theory was built in [18].

Theorem 4.18 (Steel [18])**.** $2^\omega \cap \mathrm{M}_1 = Q_3$. *Moreover, if N is a proper class inner model with a Woodin cardinal, then $Q_3 \subseteq N$.*

Definition 4.19. *Let*

$$\tilde{P}_2 = \{x \mid \exists y \in \mathrm{M}_1 (x \equiv_L y \wedge y \text{ is a master code in } \mathrm{M}_1)\}.$$

By Theorem 4.18, we have the following result.

Corollary 4.20. *For any real $x \in Q_3$, there is a real $y \in \tilde{P}_2$ so that $x \leq_L y$. Namely $\tilde{P}_2$ is L-cofinal in Q_3.*

Actually $\tilde{P}_2$ is contained in P_2.

Lemma 4.21. $\tilde{P}_2 \subseteq P_2$.

Proof. Suppose $x \in \tilde{P}_2$. Without loss of generality, x is the master code of $\mathrm{M}_1|\alpha$, where $\rho(\mathrm{M}_1|\alpha) = \omega$. Let $\mathcal{M} = \mathrm{M}_1|\alpha + 1$, so that $\rho(\mathcal{M}) = \omega$ and $x \in \mathcal{M}$. Suppose towards a contradiction that for some y, $j(x) \leq j(y)$ but $x \notin L[y]$. By Theorem 4.18, $L[y]$ does not have an inner model with a Woodin cardinal. So $K^{L[y]}$ exists. Let $\kappa = \aleph_1$. Using the fact κ is weakly compact in $L[y]$, by Schimmerling-Steel [13], $(\kappa^+)^{K^{L[y]}} = (\kappa^+)^{L[y]}$. So $x \notin L[y]$ but $(\kappa^+)^{K^{L[y]}} \geq (\kappa^+)^{L[x]}$. We shall derive a contradiction by comparing $K^{L[y]}$ versus $\mathcal{M}$.

The comparison takes place in $L[x, y]$, using L-guided iteration strategies. Both $K^{L[y]}$ and $\mathcal{M}$ are Ord $+1$-iterable in $L[x, y]$. The fact that $x \in \mathcal{M} \setminus K^{L[y]}$ implies that the $K^{L[y]}$-side comes out strictly shorter. Let $(\mathcal{T}, \mathcal{U})$ be the padded normal trees[3] on the $K^{L[y]}$-side and $\mathcal{M}$-side respectively, both of length Ord $+1$. For $\alpha \leq \beta \leq \infty$, let $\mathcal{M}^\mathcal{T}_\alpha$ be the α-th model of $\mathcal{T}$ and $i^\mathcal{T}_{\alpha\beta}$ (if exists) be the iteration map from $\mathcal{M}^\mathcal{T}_\alpha$ to $\mathcal{M}^\mathcal{T}_\beta$; similar notations apply to the $\mathcal{U}$-side. Let $\mathcal{P}$ be the last model of $\mathcal{T}$. So the iteration

[3] i.e. $E^\mathcal{T}_\alpha$ might be empty, in which case we do nothing and put $\mathcal{M}^\mathcal{T}_{\alpha+1} = \mathcal{M}^\mathcal{T}_\alpha$, and similarly for $\mathcal{U}$.

map $i^{\mathcal{T}}_{0\infty} : K^{L[y]} \to \mathcal{P}$ exists, while the main branch of $\mathcal{U}$ drops. Usual arguments (e.g. [19, Section 3]) show that there is an $L[x,y]$-definable closed unbounded proper class Γ such that for every $\xi \in \Gamma$,

(1) ξ belongs to the main branches of $\mathcal{T}$ and $\mathcal{U}$,
(2) $i^{\mathcal{T}}_{0\xi}(\xi) = \xi$, $i^{\mathcal{T}}_{\xi\infty} \upharpoonright (\xi+1) = id$,
(3) there is no drop on the main branch of $\mathcal{U}$ in the interval $[\xi, \infty)$, i.e., $i^{\mathcal{U}}_{\xi\infty}$ exists,
(4) $i^{\mathcal{U}}_{\xi\infty} \upharpoonright \xi = id$,
(5) if $\bar{\xi} < \xi$, then $o(\mathcal{M}^{\mathcal{U}}_{\bar{\xi}}) < \xi$,
(6) $\mathcal{M}^{\mathcal{T}}_{\xi} | (\xi^+)^{\mathcal{M}^{\mathcal{T}}_{\xi}} \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\xi}$.

Obviously, Γ contains every (x,y)-indiscernible, and in particular, $\kappa \in \Gamma$.

Consider the set A, where $z \in A$ iff z codes $(\mathcal{S}_z, \mathcal{N}_z, \alpha_z)$, $\mathcal{S}_z$ is a countable L-guided normal iteration tree on $\mathcal{M}$, $\mathcal{N}_z$ is the last model of $\mathcal{S}_z$, α_z is an ordinal in $\mathcal{N}_z$. A is $\Sigma^1_2(x)$, equipped with the $\Sigma^1_2(x)$ wellfounded relation $<^*$ defined as follows: $z <^* z'$ iff $\mathcal{S}_z$ is an initial segment of $\mathcal{S}_{z'}$, $\mathcal{N}_z$ is on the branch of $\mathcal{S}_{z'}$ leading from $\mathcal{M}$ to $\mathcal{N}_{z'}$, there is no drop on the branch of $\mathcal{S}_{z'}$ from $\mathcal{N}_z$ to $\mathcal{N}_{z'}$, and letting $k : \mathcal{N}_z \to \mathcal{N}_{z'}$ be the iteration map encoded in $\mathcal{S}_{z'}$, then $k(\alpha_z) > \alpha_{z'}$. Every $\Sigma^1_2(x)$ subset of ω^ω is ω_1-Suslin as witnessed by a tree on $\omega \times \omega_1$ in $L[x]$. By Kunen-Martin, the rank of $<^*$ is smaller than $j(x)$. By definition, $o(\mathcal{M}^{\mathcal{U}}_{\kappa})$ is smaller than the rank of $<^*$, hence smaller than $j(x)$.

However, the main branch of $\mathcal{T}$ does not drop, $i^{\mathcal{T}}_{0\kappa}(\kappa) = \kappa$, and $\mathcal{M}^{\mathcal{T}}_{\kappa} | (\kappa^+)^{\mathcal{M}^{\mathcal{T}}_{\kappa}} \trianglelefteq \mathcal{M}^{\mathcal{U}}_{\kappa}$, implying that $(\kappa^+)^{K^{L[y]}} \leq (\kappa^+)^{\mathcal{M}^{\mathcal{T}}_{\kappa}} \leq o(\mathcal{M}^{\mathcal{U}}_{\kappa}) < j(x)$, a contradiction. □

By Lemma 4.21, we have the following theorem.

Theorem 4.22. (1) *$\tilde{P}_2 \subseteq NCR_L$.*
(2) *NCR_L is L-cofinal in Q_3.*
(3) *NCR_L is not Σ^1_3.*

Proof. (1) follows from Lemma 4.17 and 4.21.

(2) follows from (1) and Corollary 4.20.

For (3), suppose that NCR_L is Σ^1_3. Then by (2), $Q_3 = \{x \mid \exists y \in NCR_L(x \in L[y])\}$. So Q_3 is a Σ^1_3-set, a contradiction. □

We don't know whether the converse of Lemma 4.21 is true.

Conjecture 4.23. $P_2 = \tilde{P}_2$.

Though a number of results concerning the structure of NCR_L are proved in this paper, the picture of NCR_L still remains vague for us.

Question 4.24. *Assuming $ZFC + PD$, give a clearer description of NCR_L?*

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